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LETTER TO THE EDITOR

Gauge invariance and the thermodynamics of the electromagnetic field

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Abstract. Recent discussions of the thermodynamics of the electromagnetic field interacting with two-level atoms demonstrate the existence of a phase transition. It is argued however that this only holds for field theories that do not satisfy the principle of gauge invariance.

The possibility of a phase transition in the electrodynamics of two-level atomic systems has been discussed recently by several authors, and it seems to be agreed that within the Dicke model of super-radiance one finds a phase transition provided that the number of photon modes is held fixed (Hepp and Lieb 1973a, b, Wang and Hioe 1973, Rzążewski *et al* 1975, Rzążewski and Wódkiewicz 1976). Rzążewski and his collaborators however have shown that the appearance of the phase transition in the Dicke model is due entirely to the neglect of the A^2 term from the interaction Hamiltonian ($\boldsymbol{A}$ = the Coulomb gauge vector potential), and that the inclusion of this term also serves to eliminate certain infrared divergence difficulties noted in the earlier papers cited above (Rzążewski *et al* 1975, Rzążewski and Wódkiewicz 1976).

The model of a two-level atom coupled to the electromagnetic field, which is obtained from the ordinary Hamiltonian of Coulomb gauge electrodynamics by the neglect of the e^2 term in the interaction Hamiltonian (1),

$$V^{(1)} = -\sum_{i=1}^{n} (e_i/m_i)\boldsymbol{p}_i\cdot\boldsymbol{A}(\boldsymbol{R}_i) + \sum_{i=1}^{n} (e_i^2/2m_i)A(\boldsymbol{R}_i)^2, \qquad (\mathrm{div}\,\boldsymbol{A}(\boldsymbol{x}) = 0), \tag{1}$$

has proved extremely valuable in the development of laser theory (Stenholm 1973). It must not be forgotten however that the omission of the A^2 term is usually justified only because many processes have selection rules associated with them that cause the contribution of the A^2 term to be identically zero. The purpose of this letter is to point out the obvious fact that the A^2 term appears even in the electric dipole (long-wavelength) approximation in non-relativistic electrodynamics in the Coulomb gauge because of the requirements of *gauge invariance.* The A^2 term makes a non-zero contribution to the free energy of the electromagnetic field interacting with two-level atoms; these extra terms are sufficient to eliminate the phase transition and also regularize the long-wavelength behaviour of the coupling constant. Hence the significance of the findings of Rzążewski *et al* (1975, 1976) in my view is that the existence of a phase transition and an infrared divergence has only been demonstrated in the thermodynamics of field theories that do not satisfy the principle of gauge invariance.

It is important to recognize that these remarks refer to the *Coulomb gauge* theory; if, for example, one makes the usual Fourier expansion of the Coulomb gauge vector potential, the interaction term in the model Hamiltonian is proportional to $(1/\sqrt{w})$ where w is the mode frequency, and it is this frequency dependence that leads to the infrared difficulty noted by Hepp and Lieb (1973b). As is now well known it is possible to obtain a new Hamiltonian in which the interaction term in the electric dipole approximation is simply,

$$V^{(2)} = -\boldsymbol{d} \cdot \boldsymbol{E}^{\perp} \qquad (2)$$

where $\boldsymbol{d}$ is the atomic dipole moment operator, and $\boldsymbol{E}^{\perp}$ is the transverse electric field strength (Power and Zienau 1959, Woolley 1975). In this representation, which is equivalent to working in a different gauge, an 'A^2' term can only contribute when one goes *beyond* the electric dipole approximation, and this feature makes the interaction Hamiltonian (2) computationally easier to work with; there are many explicit demonstrations in the literature of the equivalence on the energy shell of the Coulomb gauge interaction Hamiltonian including the A^2 term (1), in the electric dipole approximation, and the dipole interaction (2) (Power 1964). As a simple example of how the single interaction term (2) differs from the truncated model interaction based on (1), it may be noted that the Fourier expansion of $\boldsymbol{E}(\boldsymbol{x})^{\perp}$ is proportional to $\sqrt{w}$, and this regularizes the long-wavelength behaviour of the effective coupling constant that appears in the thermodynamics of the electromagnetic field. Hence one can expect a calculation based on the interaction Hamiltonian (2) to be in agreement with the calculations of Rzążewski *et al* (1975, 1976).

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